

Invariant Gibbs measures of the energy for shell models of turbulence; the inviscid and viscous cases

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Abstract

Gaussian measures of Gibbsian type are associated with some shell model of 3D turbulence; they are constructed by means of the energy, a conserved quantity for the 3D inviscid and unforced shell model. We prove the existence of a unique global flow for a stochastic viscous shell model and of a global flow for the deterministic inviscid shell model, with the property that these Gibbs measures are invariant for these flows.

1 Introduction

The study of existence and uniqueness of solutions for incompressible inviscid and viscous flows with initial data in some "physically relevant" space is of great interest. The most understood case is the 2D model, for which existence and uniqueness of classical and weak solutions for the viscous flow with initial data of finite energy are due to J. Leray and later to O. Ladyzenskaja, J.-L. Lions and G. Prodi, while for the inviscid flow they are due to W. Wolibner and later to V. Yudovich, T. Kato and C. Bardos with more assumptions on the initial velocity. However, the 3D motion is a more challenging problem; for the viscous case, Leray's work gives existence but not uniqueness of weak solutions for initial data of finite energy, whereas with more restrictive assumptions on the initial data there exists a unique local solution. For the 3D inviscid case, only local results for the well posedness of weak solutions are known. We refer to [25] where the authors, C. Marchioro and M. Pulvirenti provide a comprehensive introduction to a wide range of topics related to equations of inviscid and incompressible fluid flow. The interested reader can find a quite recent account of all these results in [23].

Inside the analysis of the equations of hydrodynamics, statistical solutions have been investigated. In fact the individual solutions may give a detailed and too complicated picture of the fluid, while one could be interested in the behavior of some global quantity related to the fluid, where the microscopic picture is

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replaced by the macroscopic one. This is the statistical approach to turbulence. From the mathematical point of view, we are interested in distributions invariant for these flows. Probability measures of Gibbsian type, with Gibbs density expressed by means of invariants of the 2D motions have been discussed in [1, 6, 3, 4, 5, 10, 8, 11, 13, 16, 18, 2]. The construction of such invariants measures looks quite natural since the 2D Euler equation has the nice property to admit infinitely many first integrals, including the quadratic invariants given by the energy and the enstrophy. In particular, in all the previous papers the Gibbs density is constructed by means of the enstrophy. No results of Gibbs invariant measures are known for the 3D equations of hydrodynamics.

In this paper, we consider some shell models in a very general form which includes the SABRA and the GOY models. These models are the most interesting and most popular examples of simplified phenomenological models of turbulence. This is because, although departing from reality, they capture some essential statistical properties and features of turbulent flows, like the energy and the enstrophy cascade and the power law decay of the structure functions in some range of wave numbers, the inertial range. From the computational point of view, shell models are much simpler to simulate than the Navier-Stokes equations due to the fact that we need a moderate number of degrees of freedom to reach high Reynolds numbers (see, e.g., [24] and references therein). Indeed, shell models of turbulence describe the evolution of complex Fourier-like components of a scalar velocity field denoted by u_n and the associated wavenumbers are denoted by k_n , where the discrete index n is referred as the shell index. The evolution of the infinite sequence $\{u_n\}_{n=-1}^{\infty}$ is given by

$$(1) \quad \dot{u}_n(t) + \nu k_n^2 u_n(t) + b_n(u(t), u(t)) = f_n(t, u(t)), \quad n = 1, 2, \dots$$

with $u_{-1} = u_0 = 0$ and $u_n(t) \in \mathbb{C}$ for $n \geq 1$. Here $\nu \geq 0$ and in analogy with Navier-Stokes equations ν represents a kinematic viscosity; $k_n = k_0 \lambda^n$ ($\lambda > 1$) and f_n is a forcing term. The exact form of $b_n(u, v)$ varies from one model to another. However in all the various models, it is assumed that $b_n(u, v)$ is chosen in such a way that

$$(2) \quad \Re \sum_{n=1}^{\infty} b_n(u, v) \bar{v}_n = 0,$$

where \Re denotes the real part and \bar{x} the complex conjugate of x . Equation (2) implies a formal law of conservation of energy in the inviscid ($\nu = 0$) and unforced form of (1). These models have similar properties to 3D fluids.

In particular, we define the bilinear terms b_n as

$$b_n(u, v) = i(ak_{n+1}\bar{u}_{n+1}\bar{v}_{n+2} + bk_n\bar{u}_{n-1}\bar{v}_{n+1} - ak_{n-1}\bar{u}_{n-1}\bar{v}_{n-2} - bk_{n-1}\bar{u}_{n-2}\bar{v}_{n-1})$$

in the GOY model (see [20, 26]) and by

$$b_n(u, v) = -i(ak_{n+1}\bar{u}_{n+1}v_{n+2} + bk_n\bar{u}_{n-1}v_{n+1} + ak_{n-1}u_{n-1}v_{n-2} + bk_{n-1}u_{n-2}v_{n-1})$$

in the SABRA model (see [24]).

The two parameters a, b are real numbers.

In the present paper, we consider particular Gaussian measures of Gibbs type and investigate their role in the analysis of shell models. Basically, these

Gibbs measures μ^ν are constructed by means of the energy, which is an invariant of motion for the inviscid and unforced shell model. Therefore, our aim is to show that these measures are invariant for the inviscid shell model as well as for a suitable stochastic viscous shell model. The support of the measure μ^ν is a Sobolev space of negative exponent and the space of finite energy initial velocity is negligible with respect to the measure μ^ν . Thus, one looks for a flow with initial data of infinite energy.

Our results are very similar to those proved for the 2D stochastic Navier–Stokes and 2D deterministic Euler equation with respect to the Gibbs measure of the enstrophy (a conserved quantity for the 2D equation of hydrodynamics) in a series of papers [1, 6, 3, 4, 5, 13, 16, 18, 2]. However, our results hold for general shell models for which only the energy is an invariant of motion and are therefore approximation models for 3D hydrodynamics.

Let us describe the content of the paper in more details. In Section 2 we introduce the equations, the Gibbs measure μ^ν and their basic properties; in particular, we introduce the Ornstein-Uhlenbeck equation with a suitable noise and prove that the Gibbs measure μ^ν is its unique invariant measure. In Section 3 we focus on the stochastic viscous shell model, having μ^ν as invariant measure; first, we prove that for μ^ν -a.e. initial data there exists a unique global solution, and then that there exists a unique stationary process whose law at any fixed time is μ^ν . The last Section 4 deals with the inviscid shell model, for which we prove that there exists a stationary process solving it and whose law at any fixed time is μ^ν .

2 Functional setting

Even if in (1) we considered the unknowns $u_n(t) \in \mathbb{C}$, from now on we deal with the real part and the imaginary part of u_n : $u_{n,1} = \Re u_n$ and $u_{n,2} = \Im u_n$. As usual, for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $|x|^2 = x_1^2 + x_2^2$ and $x \cdot y = x_1 y_1 + x_2 y_2$ is the scalar product in \mathbb{R}^2 .

2.1 Spaces and operators

For any $\alpha \in \mathbb{R}$ set

$$H^\alpha = \{u = (u_1, u_2, \dots) \in (\mathbb{R}^2)^\infty : \sum_{n=1}^{\infty} k_n^{2\alpha} |u_n|^2 < \infty\}.$$

This is a Hilbert space with scalar product $\langle u, v \rangle_{H^\alpha} = \sum_{n=1}^{\infty} k_n^{2\alpha} u_n \cdot v_n$. Denote by $\|\cdot\|_{H^\alpha}$ its norm. We have the continuous embedding

$$H^{\alpha_1} \subset H^{\alpha_2} \quad \text{if } \alpha_1 > \alpha_2.$$

Let A be the linear unbounded operator in H^0 defined as

$$A : (u_1, u_2, \dots) \mapsto (k_1^2 u_1, k_2^2 u_2, \dots), \quad D(A) = H^2.$$

The fractional power operators A^p are well defined for any $p \in \mathbb{R}$:

$$A^p : H^{2p+\beta} \rightarrow H^\beta, \quad (u_1, u_2, \dots) \mapsto (k_1^{2p} u_1, k_2^{2p} u_2, \dots)$$

in any space (i.e. for any β). For any $p < 0$, A^p is a trace class operator in H^β , since $\text{Tr}(A^p) = \sum_n k_n^{2p} = k_0^{2p} \sum_n \lambda^{2pn}$ is finite if and only if $p < 0$; therefore, the operator A^p ($p < 0$) is compact and Hilbert–Schmidt as a linear operator in $H^{2p+\beta}$.

Moreover, A generates an analytic semigroup of contractions in H^0 and for any $p > 0$ and $t > 0$

$$(3) \quad \|A^p e^{-\nu A t} x\|_{H^0} \leq \frac{c_{p,\nu}}{t^p} \|x\|_{H^0},$$

with $c_{p,\nu} = (\frac{p}{e\nu})^p$.

Set $B_n = (B_{n,1}, B_{n,2})$, where $B_{n,1}$ and $B_{n,2}$ are, respectively, the real part and the imaginary part of the b_n given in the previous section. For instance, in the SABRA model

$$(4) \quad \begin{aligned} B_{1,1}(u, v) &= ak_2[-u_{2,2}v_{3,1} + u_{2,1}v_{3,2}] \\ B_{1,2}(u, v) &= -ak_2u_2 \cdot v_3 \end{aligned}$$

$$(5) \quad \begin{aligned} B_{2,1}(u, v) &= ak_3[-u_{3,2}v_{4,1} + u_{3,1}v_{4,2}] + bk_2[-u_{1,2}v_{3,1} + u_{1,1}v_{3,2}] \\ B_{2,2}(u, v) &= -ak_3u_3 \cdot v_4 - bk_2u_1 \cdot v_3 \end{aligned}$$

and for $n > 2$

$$(6) \quad \begin{aligned} B_{n,1}(u, v) &= ak_{n+1}[-u_{n+1,2}v_{n+2,1} + u_{n+1,1}v_{n+2,2}] \\ &\quad + bk_n[-u_{n-1,2}v_{n+1,1} + u_{n-1,1}v_{n+1,2}] \\ &\quad + ak_{n-1}[u_{n-1,2}v_{n-2,1} + u_{n-1,1}v_{n-2,2}] \\ &\quad + bk_{n-1}[u_{n-2,2}v_{n-1,1} + u_{n-2,1}v_{n-1,2}], \end{aligned}$$

$$(7) \quad \begin{aligned} B_{n,2}(u, v) &= -ak_{n+1}[u_{n+1,1}v_{n+2,1} + u_{n+1,2}v_{n+2,2}] \\ &\quad - bk_n[u_{n-1,1}v_{n+1,1} + u_{n-1,2}v_{n+1,2}] \\ &\quad - ak_{n-1}[u_{n-1,1}v_{n-2,1} - u_{n-1,2}v_{n-2,2}] \\ &\quad - bk_{n-1}[u_{n-2,1}v_{n-1,1} - u_{n-2,2}v_{n-1,2}]. \end{aligned}$$

Define the bilinear operator $B : (\mathbb{R}^2)^\infty \times (\mathbb{R}^2)^\infty \rightarrow (\mathbb{R}^2)^\infty$ as

$$B(u, v) = (B_1(u, v), B_2(u, v), \dots).$$

We have that B is well defined when its domain is $H^1 \times H^0$ or $H^1 \times H^0$ (see [14]), that is $B : H^1 \times H^0 \rightarrow H^0$ and $B : H^1 \times H^0 \rightarrow H^0$ are bounded operators. We extend the result of [14] to more general spaces; this is very similar to Proposition 1 of [15].

Lemma 2.1 *For any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$*

$$B : H^{\alpha_1} \times H^{\alpha_2} \rightarrow H^{-\alpha_3} \quad \text{with } \alpha_1 + \alpha_2 + \alpha_3 = 1$$

and there exists a constant c (depending on a, b, λ and the α_j 's) such that

$$\|B(u, v)\|_{H^{-\alpha_3}} \leq c \|u\|_{H^{\alpha_1}} \|v\|_{H^{\alpha_2}} \quad \forall u \in H^{\alpha_1}, v \in H^{\alpha_2}.$$

Proof. The proof is similar to that of Proposition 1 in [14]. We write it for reader's convenience.

First,

$$\|B(u, v)\|_{H^{-\alpha_3}} = \sup_{\|z\|_{H^{\alpha_3}} \leq 1} \left| \sum_{n=1}^{\infty} B_n(u, v) \cdot z_n \right|.$$

Now we estimate the trilinear term. Looking at the expression of the B_n 's we have eight series $\sum_{n=1}^{\infty}$ to consider. We write the details for the first one, working on the others in the same way.

$$\begin{aligned}
(8) \quad & \sum_{n=1}^{\infty} |ak_{n+1}u_{n+1,2}v_{n+2,1}z_{n,1}| \\
&= \sum_{n=1}^{\infty} ak_0\lambda^{n+1}|u_{n+1,2}v_{n+2,1}z_{n,1}| \\
&= a\lambda^{1-\alpha_1-2\alpha_2} \sum_{n=1}^{\infty} (k_0\lambda^{n+1})^{\alpha_1} |u_{n+1,2}| (k_0\lambda^{n+2})^{\alpha_2} |v_{n+2,1}| (k_0\lambda^n)^{\alpha_3} |z_{n,1}| \\
&= a\lambda^{1-\alpha_1-2\alpha_2} \sum_{n=1}^{\infty} k_{n+1}^{\alpha_1} |u_{n+1,2}| k_{n+2}^{\alpha_2} |v_{n+2,1}| k_n^{\alpha_3} |z_{n,1}| \\
&\leq a\lambda^{1-\alpha_1-2\alpha_2} \sqrt{\sum_{n=1}^{\infty} k_{n+1}^{2\alpha_1} |u_{n+1,2}|^2} \sqrt{\sum_{n=1}^{\infty} k_{n+2}^{2\alpha_2} |v_{n+2,1}|^2} \sup_n (k_n^{\alpha_3} |z_{n,1}|) \\
&\leq a\lambda^{1-\alpha_1-2\alpha_2} \|u\|_{H^{\alpha_1}} \|v\|_{H^{\alpha_2}} \|z\|_{H^{\alpha_3}}.
\end{aligned}$$

□

In particular, B is a bounded operator in the following spaces:

$$(9) \quad B : H^{-\alpha} \times H^{-\alpha} \rightarrow H^{-2\alpha-1},$$

$$(10) \quad B : H^{-\alpha} \times H^0 \rightarrow H^{-\alpha-1}, \quad B : H^0 \times H^{-\alpha} \rightarrow H^{-\alpha-1}$$

and

$$(11) \quad B : H^{-2-\alpha} \times H^{-2-\alpha} \rightarrow H^{-5-2\alpha}$$

for $\alpha \in \mathbb{R}$.

A remarkable property of the operator B is

$$(12) \quad \sum_{n=1}^{\infty} B_n(u, v) \cdot v_n = 0$$

whenever u and v give sense to the l.h.s..

2.2 Gibbs measure of the energy

For any $\nu > 0$, let us define the probability measure μ^ν on $(\mathbb{R}^2)^\infty$ as

$$\mu^\nu = \otimes_{n=1}^{\infty} \mu_n^\nu,$$

where μ_n^ν is the Gaussian measure on \mathbb{R}^2 :

$$\mu_n^\nu(du_n) = \frac{\nu}{2\pi} e^{-\nu \frac{1}{2}(u_{n,1}^2 + u_{n,2}^2)} du_{n,1} du_{n,2}.$$

Heuristically we have

$$\mu^\nu(du) = \frac{1}{Z} e^{-\nu E(u)} du'',$$

where $E = \frac{1}{2} \sum_{n=1}^{\infty} |u_n|^2$ is the energy and Z is a normalization constant to make μ^ν to be a probability measure. This is the reason to call μ^ν the Gibbs measure of the energy with parameter ν .

The support of the measure μ^ν is bigger than the space H^0 of finite energy. Indeed, for any $c > 0$ we have $\mu^\nu(\{x \in (\mathbb{R}^2)^\infty : \sup_n |x_n| < c\}) = 0$; since the space H^0 is contained in the space of bounded sequences, we have also that

$$\mu^\nu(H^0) = 0.$$

Moreover

$$\begin{aligned} \int \|u\|_{H^\alpha}^2 \mu^\nu(du) &= \sum_{n=1}^{\infty} k_n^{2\alpha} \iint_{\mathbb{R}^2} (u_{n,1}^2 + u_{n,2}^2) \frac{\nu}{2\pi} e^{-\nu \frac{1}{2}(u_{n,1}^2 + u_{n,2}^2)} du_{n,1} du_{n,2} \\ &= \frac{2}{\nu} \sum_{n=1}^{\infty} k_n^{2\alpha} = \frac{2}{\nu} k_0^{2\alpha} \sum_{n=1}^{\infty} \lambda^{2\alpha n}. \end{aligned}$$

This is finite if and only if $\alpha < 0$. Hence,

$$\mu^\nu(H^\alpha) = 1 \quad \forall \alpha < 0.$$

Thus we set

$$\mathbb{H} = \cap_{\alpha < 0} H^\alpha.$$

\mathbb{H} is a Fréchet space (see, e.g., [28]) and

$$\mu^\nu(\mathbb{H}) = 1.$$

According to Kakutani's theorem (see [22]), the measures μ^{ν_1} and μ^{ν_2} are orthogonal (i.e., mutually singular) for $\nu_1 \neq \nu_2$ positive.

For $p \geq 1$ we denote by $\mathcal{L}_{\mu^\nu}^p$ the space of Borelian functions $\phi : \mathbb{H} \rightarrow \mathbb{R}$ such that $\int |\phi|^p d\mu^\nu < \infty$. We have $\mathcal{L}_{\mu^\nu}^p \subseteq \mathcal{L}_{\mu^\nu}^q$ for $p \geq q$.

Let FC_b^∞ be the space of infinitely differentiable cylindrical functions bounded and with bounded derivatives, that is

$$\exists k \in \mathbb{N} : \phi = \phi(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \in C_b^\infty((\mathbb{R}^2)^k; \mathbb{R}).$$

Analogously let $FPol$ be the space of cylindrical polynomial functions. Either FC_b^∞ or $FPol$ is a dense subspace of $\mathcal{L}_{\mu^\nu}^p$ for $1 \leq p < \infty$.

An important property is the integrability of B with respect to the measure μ^ν .

Proposition 2.2

$$\int \|B(x, x)\|_{H^{-1-\alpha}}^p \mu^\nu(dx) < \infty$$

for any $p \in \mathbb{N}$ and $\alpha > 0$.

Proof. We write the proof for $p = 2$ but it is the same for the other values of p , since μ^μ is Gaussian and the B_n 's are second order polynomial. The details are given for the SABRA model, but the result is true for all "finite" shell models.

We have

$$\begin{aligned}
\int |B_{n,1}(x, x)|^2 \mu^\nu(dx) &= \int |ak_{n+1}[-x_{n+1,2}x_{n+2,1} + x_{n+1,1}x_{n+2,2}] \\
&\quad + bk_n[-x_{n-1,2}x_{n+1,1} + x_{n-1,1}x_{n+1,2}] \\
&\quad + (a+b)k_{n-1}[x_{n-1,2}x_{n-2,1} + x_{n-1,1}x_{n-2,2}]|^2 \mu^\nu(dx) \\
&\leq 2 \int \{a^2k_{n+1}^2[x_{n+1,2}^2x_{n+2,1}^2 + x_{n+1,1}^2x_{n+2,2}^2] \\
&\quad + b^2k_n^2[x_{n-1,2}^2x_{n+1,1}^2 + x_{n-1,1}^2x_{n+1,2}^2] \\
&\quad + (a+b)^2k_{n-1}^2[x_{n-1,2}^2x_{n-2,1}^2 + x_{n-1,1}^2x_{n-2,2}^2]\} \mu^\nu(dx) \\
&= \frac{16}{\nu^2} \{a^2k_{n+1}^2 + b^2k_n^2 + (a+b)^2k_{n-1}^2\} \\
&= \frac{16}{\nu^2} k_0^2 \{a^2\lambda^4 + b^2\lambda^2 + (a+b)^2\} \lambda^{2(n-1)}.
\end{aligned}$$

Similarly we estimate $\int |B_{n,2}(x, x)|^2 \mu^\nu(dx)$. Therefore

$$\begin{aligned}
\int \|B(x, x)\|_{H^{-1-\alpha}}^2 \mu^\nu(dx) &= \int \sum_{n=1}^{\infty} k_n^{2(-1-\alpha)} |B_n(x, x)|^2 \mu^\nu(dx) \\
&\leq c_{\nu, k_0, \lambda} (|a|^2 + |b|^2) \sum_{n=1}^{\infty} \lambda^{-2n\alpha}
\end{aligned}$$

which is finite if and only if $\alpha > 0$. □

We give a definition.

Definition 2.3 We say that a process $v = \{v_t\}_{t \geq 0}$ is a μ^ν -stationary process if
i) v is a stationary process;
ii) the law of $v(t)$ is μ^ν for any $t \geq 0$.

Notice that, if v is a μ^ν -stationary process defined on $(\Omega, \mathbb{F}, \mathbb{P})$, denoting by \mathbb{E} the mathematical expectation we have

$$\begin{aligned}
(13) \quad \mathbb{E} \left[\int_0^T \|B(v(t), v(t))\|_{H^{-1-\alpha}}^p dt \right] &= \int_0^T \mathbb{E} [\|B(v(t), v(t))\|_{H^{-1-\alpha}}^p] dt \\
&= T \int_{\mathbb{H}} \|B(x, x)\|_{H^{-1-\alpha}}^p \mu^\nu(dx) < \infty.
\end{aligned}$$

Therefore, we have the following result.

Proposition 2.4 Let v be a μ^ν -stationary process. Then, for any $p \geq 1, \alpha < 0$ we have that $B(v, v) \in L^p(0, T; H^{-1-\alpha})$ \mathbb{P} -a.s..

2.3 The Wiener process

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space, with expectation denoted by \mathbb{E} . Consider a sequence $\{w_{n,j}\}_{n \in \mathbb{N}; j=1,2}$ of independent standard 1-dimensional

Brownian motions defined for all real t . We say that w is an H^0 -cylindrical Wiener process if

$$w = ((w_{1,1}, w_{1,2}), (w_{2,1}, w_{2,2}), (w_{3,1}, w_{3,2}), \dots).$$

We set $\mathbb{F}_t = \sigma\{w(s_2) - w(s_1), s_1 \leq s_2 \leq t\}$.

The paths of the process w are (\mathbb{P} -a.s.) in $C^\beta([t_0, T]; H^{-\alpha})$ for any $-\infty < t_0 < T < +\infty$, $0 \leq \beta < \frac{1}{2}$ and $\alpha > 0$. In fact, $w(t)$ has values in $H^{-\alpha}$ for any $\alpha > 0$ since

$$\mathbb{E}[\|w(t)\|_{H^{-\alpha}}^2] = \mathbb{E}\left[\sum_{n=1}^{\infty} k_n^{-2\alpha} |w_n(t)|^2\right] = 2|t| \sum_{n=1}^{\infty} k_n^{-2\alpha} = 2k_0^{-2\alpha} |t| \sum_{n=1}^{\infty} \lambda^{-2\alpha n},$$

and the latter series converges if and only if $\alpha > 0$.

Moreover, with similar argument we get from Kolmogorov criterium (see, e.g., [17] Theorem 3.3) that

$$(14) \quad \mathbb{E}[\|w\|_{C^\beta([t_0, T]; H^{-\alpha})}^2] < \infty \quad \text{for any } \beta \in [0, \frac{1}{2}), \alpha > 0.$$

2.4 The equations

Let us consider the stochastic viscous shell model

$$(15) \quad du(t) + [\nu Au(t) + B(u(t), u(t))]dt = \sqrt{2A} dw(t).$$

As we shall see in the next section, the covariance of the Wiener process has been chosen in such a way that the measure μ^ν is invariant for (15) (in a sense to be specified later on); with this type of covariance we cannot analyze equation (15) with classical techniques, as done for instance in [7].

When there is no viscosity and forcing term in (15), we get the deterministic unforced and inviscid shell model

$$(16) \quad \frac{du}{dt}(t) + B(u(t), u(t)) = 0.$$

From property (12) we have that the energy $E(t) = \frac{1}{2} \sum_{n=1}^{\infty} |u_n(t)|^2$ is an invariant of motion for (16), i.e. for any t

$$\frac{dE}{dt}(t) = \sum_{n=1}^{\infty} \dot{u}_n(t) \cdot u_n(t) = - \sum_{n=1}^{\infty} B_n(u(t), u(t)) \cdot u_n(t) = 0,$$

whenever we consider a dynamics giving sense to the latter quantities.

We are interested also in the linear stochastic equation, i.e. the Ornstein–Uhlenbeck equation

$$(17) \quad dz(t) + \nu Az(t) dt = \sqrt{2A} dw(t).$$

For any time interval $[t_0, T]$, this equation has a unique strong solution

$$(18) \quad z(t) = e^{-\nu(t-t_0)A} z(t_0) + \int_{t_0}^t e^{-\nu(t-s)A} \sqrt{2A} dw(s).$$

This is easy to prove for this linear stochastic equation, which corresponds to an infinite system of decoupled linear equations ($n \in \mathbb{N}$, $j = 1, 2$)

$$(19) \quad dz_{n,j}(t) + \nu k_n^2 z_{n,j}(t) dt = \sqrt{2} k_n dw_{n,j}(t).$$

We have that $z(t)$ takes values in $H^{-\alpha}$ ($\alpha > 0$) \mathbb{P} -a.s. if $z(t_0) \in H^{-\alpha}$. Indeed, if $z(t_0)$ is in $H^{-\alpha}$ then $e^{-\nu(t-t_0)A} z(t_0)$ stays in the same space. And for the stochastic integral we have

$$\begin{aligned} \mathbb{E}[\|\int_{t_0}^t e^{-\nu(t-s)A} \sqrt{2A} dw(s)\|_{H^{-\alpha}}^2] &= \mathbb{E} \sum_{n=1}^{\infty} \sum_{j=1}^2 k_n^{-2\alpha} |\int_{t_0}^t e^{-\nu(t-s)k_n^2} \sqrt{2} k_n dw_{n,j}(s)|^2 \\ &= \sum_{n,j} k_n^{-2\alpha} \int_{t_0}^t e^{-2\nu(t-s)k_n^2} 2k_n^2 ds \\ &\leq \frac{2}{\nu} \sum_{n=1}^{\infty} k_n^{-2\alpha} = \frac{2}{\nu} k_0^{-2\alpha} \sum_{n=1}^{\infty} \lambda^{-2\alpha n}. \end{aligned}$$

Moreover, the paths are a.s. continuous in time. In fact, the continuity of the trajectories is easily obtained, because A is a diagonal operator commuting with the covariance operator of the Wiener process w (see [17] Theorem 5.9). Further (see [17] Remark 5.11) we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \|z(t)\|_{H^{-\alpha}}^p < \infty$$

for any $p \geq 1$.

Finally, the stationary solution to (17) can be represented as

$$(20) \quad \zeta(t) = \int_{-\infty}^t e^{-\nu(t-s)A} \sqrt{2A} dw(s),$$

and the law of $\zeta(t)$ is μ^ν for any t .

2.5 Invariance of the measure μ^ν

Let us consider how the measure μ^ν is related to the three equations considered in the previous section. We present well known properties, which are similar to those for the 2D Navier-Stokes equation with respect to the Gibbs measure of the enstrophy (see [1, 2, 3, 4, 5, 6]).

We start with the easy linear stochastic case (17). Denote by $z_x(t)$ the unique strong solution of equation (17) started at time $t = 0$ from x and evaluated at time $t > 0$; this has been given in (18).

We have that the measure μ^ν is the unique invariant measure of equation (17), in the sense that

$$(21) \quad \int \mathbb{E}[\phi(z_x(t))] \mu^\nu(dx) = \int \phi(x) \mu^\nu(dx) \quad \forall t \geq 0, \phi \in \mathcal{L}_{\mu^\nu}^2.$$

Indeed, we define the Markov semigroup $\{R_t\}_{t \geq 0}$ as

$$(22) \quad (R_t \phi)(x) = \mathbb{E}[\phi(z_x(t))].$$

Hence the invariance of the measure μ^ν is

$$(23) \quad \int R_t \phi(x) \mu^\nu(dx) = \int \phi(x) \mu^\nu(dx) \quad \forall t \geq 0, \phi \in \mathcal{L}_{\mu^\nu}^2.$$

Formally, we have $R_t = e^{-tQ}$, Q being the Ornstein–Uhlenbeck operator. The concrete expression of the generator is easily given on particular dense subspaces of $\mathcal{L}_{\mu^\nu}^2$. For $\phi \in FC_b^\infty$ we have

$$(24) \quad Q\phi(x) = \sum_n \sum_{j=1}^2 \left[k_n^2 \frac{\partial^2 \phi}{\partial x_{n,j}^2}(x) - \nu k_n^2 x_{n,j} \frac{\partial \phi}{\partial x_{n,j}}(x) \right]$$

Here for $\phi : (\mathbb{R}^2)^k \rightarrow \mathbb{R}$, $\phi \in FC_b^\infty$ we set

$$\tilde{\phi}(x_{i_1,1}, x_{i_1,2}, x_{i_2,1}, x_{i_2,2}, \dots, x_{i_k,1}, x_{i_k,2}) = \phi(x_{i_1}, x_{i_2}, \dots, x_{i_k})$$

and

$$\frac{\partial \phi}{\partial x_{n,j}}(x_{i_1}, x_{i_2}, \dots) = \frac{\partial \tilde{\phi}}{\partial x_{n,j}}(x_{i_1,1}, x_{i_1,2}, x_{i_2,1}, x_{i_2,2}, \dots).$$

Hence (23) is equivalent to the infinitesimal invariance

$$(25) \quad \int Q\phi \, d\mu^\nu = 0 \quad \forall \phi \in D(Q)$$

Notice that Q is symmetric when defined on FC_b^∞ :

$$(26) \quad \int Q\phi \, \psi \, d\mu^\nu = \int \phi \, Q\psi \, d\mu^\nu \quad \forall \phi, \psi \in FC_b^\infty,$$

by direct computation, using integration by parts. Therefore the semigroup R_t is symmetric; hence we can define R_t in any space $\mathcal{L}_{\mu^\nu}^p$ ($1 \leq p \leq \infty$); first by the $\mathcal{L}_{\mu^\nu}^1 - \mathcal{L}_{\mu^\nu}^\infty$ duality we define it uniquely in $\mathcal{L}_{\mu^\nu}^1$ and then by Riesz–Thorin theorem in $\mathcal{L}_{\mu^\nu}^p$ for $1 < p < \infty$. For simplicity, let us work in the Hilbert setting of $\mathcal{L}_{\mu^\nu}^2$.

In particular, taking $\psi = 1$ in (26) we get

$$(27) \quad \int Q\phi \, d\mu^\nu = 0 \quad \forall \phi \in FC_b^\infty.$$

This implies (25) if FC_b^∞ is dense in $D(Q)$ with respect to the Q -operator norm, or equivalently if (Q, FC_b^∞) is essentially self-adjoint. The fact that FC_b^∞ is a core for the infinitesimal generator of the semigroup R_t is proved by means of Theorem X.40 of [29] since the semigroup is given by (22) with (18).

As far as the nonlinear equation (16) is concerned, we have that the measure μ^ν is infinitesimally invariant, that is

$$(28) \quad \int L\phi \, d\mu^\nu = 0 \quad \forall \phi \in FC_b^\infty,$$

where (L, FC_b^∞) is the Liouville operator associated to equation (16), that is

$$L\phi(x) = - \sum_{n,j} B_{n,j}(x, x) \frac{\partial \phi}{\partial x_{n,j}}(x).$$

It is a linear operator in $\mathcal{L}_{\mu^\nu}^2$ with dense domain FC_b^∞ and is skew-symmetric, i.e.

$$(29) \quad \int L\phi \psi \, d\mu^\nu = - \int \phi L\psi \, d\mu^\nu \quad \forall \phi, \psi \in FC_b^\infty.$$

Indeed, integrating by parts and noting that each $B_{n,j}$ does not depend on the variable $x_{n,j}$, we have

$$\begin{aligned} \int L\phi \psi \, d\mu^\nu &= - \int \sum_{n,j} B_{n,j}(x, x) \frac{\partial \phi}{\partial x_{n,j}}(x) \psi(x) \mu^\nu(dx) \\ &= -\nu \underbrace{\int \sum_{n,j} B_{n,j}(x, x) x_{n,j} \phi(x) \psi(x) \mu^\nu(dx)}_{= 0 \text{ by (12)}} \\ &\quad + \int \phi(x) \sum_{n,j} B_{n,j}(x, x) \frac{\partial \psi}{\partial x_{n,j}}(x) \mu^\nu(dx) = - \int \phi L\psi \, d\mu^\nu. \end{aligned}$$

Taking $\psi = 1$ in (29), we get (28).

Actually, if $\phi \in FC_b^1$ then $L\phi \in \mathcal{L}_{\mu^\nu}^q$ for any $q \geq 1$, since the $B_{n,j}$'s are polynomials and μ^ν is a Gaussian measure.

As far as the nonlinear stochastic equation (15) is concerned, its Kolmogorov operator is given by

$$\begin{aligned} K\phi(x) &= \sum_n \sum_{j=1}^2 \left[k_n^2 \frac{\partial^2 \phi}{\partial x_{n,j}^2}(x) - B_{n,j}(x, x) \frac{\partial \phi}{\partial x_{n,j}}(x) - \nu k_n^2 x_{n,j} \frac{\partial \phi}{\partial x_{n,j}}(x) \right] \\ &\equiv (Q + L)\phi \end{aligned}$$

for $\phi \in FC_b^\infty$. Again, given $\phi \in FC_b^2$ we have that $K\phi \in \mathcal{L}_{\mu^\nu}^q$ for any $q \geq 1$. Hence we can consider K as a linear operator in $\mathcal{L}_{\mu^\nu}^q$ with dense domain FC_b^∞ .

Since $K = Q + L$, from (27) and (28) follows that the measure μ^ν is infinitely invariant for K , that is

$$(30) \quad \int K\phi \, d\mu^\nu = 0 \quad \forall \phi \in FC_b^\infty.$$

3 Stochastic viscous shell models

Let us consider equation (15). We are interested in solutions u_x with initial data $x \in \mathbb{H}$. To this end, we consider initial data in $H^{-\alpha}$ for $0 < \alpha < 1$. We assume in the whole section that the parameters $\nu > 0$ and $\alpha \in]0, 1[$ are fixed. The results hold true for arbitrary values of ν and α in the given range.

Here is our main result.

Theorem 3.1 *There exists a set $\tilde{H} \subset \mathbb{H}$ with $\mu^\nu(\tilde{H}) = 1$ such that for any $x \in \tilde{H} \cap H^{-\alpha}$ there exists a unique strong solution u_x to equation (15) with $u(0) = x$ and*

$$(31) \quad \mathbb{E} \|u_x\|_{C([0,T]; H^{-\alpha})}^p < \infty$$

for any $p \geq 1$ and any finite time interval $[0, T]$.

Moreover, the measure μ^ν is invariant for equation (15) in the sense that

$$(32) \quad \int \mathbb{E}\phi(u_x(t)) \mu^\nu(dx) = \int \phi(x) \mu^\nu(dx)$$

for any $t \geq 0$ and $\phi \in \mathcal{L}_{\mu^\nu}^1(H^{-\alpha})$.

We prove this result in two steps, following the lines of [18]. First, we construct a unique local solution. Then, by means of an approximating problem we obtain an a priori estimate, which provides the global existence.

Remark 3.2 *The solution u_x is a strong solution in the probabilistic sense and is pathwise unique. It allows to construct the Markov semigroup $\{P_t\}_{t \geq 0}$ as*

$$(P_t\phi)(x) = \mathbb{E}[\phi(u_x(t))].$$

We have $P_t : B_b(H^{-\alpha}) \rightarrow B_b(H^{-\alpha})$ for any $t \geq 0$, where $B_b(H^{-\alpha})$ is the space of Borel bounded functions $\phi : H^{-\alpha} \rightarrow \mathbb{R}$. Actually, $P_t\phi(x)$ is defined only for μ^ν -a.e. x , but this is enough to give sense to (32). In particular, thanks to (31) P_t is well defined on polynomial functions with values in $\mathcal{L}_{\mu^\nu}^q$ for any $1 \leq q < \infty$; hence P_t can be extended to $\mathcal{L}_{\mu^\nu}^q$.

3.1 Local existence

Consider equation (15) with initial data $u(0) = x \in H^{-\alpha}$. We prove that it has a unique local solution, where the time interval on which it is defined is random. This basically relies on the fact that the equation has a locally Lipschitz nonlinearity. To this end, let us deal with the solution of equation (15) in the mild form (see, e.g., [17])

$$(33) \quad u_x(t) = e^{-\nu t A} x - \int_0^t e^{-\nu(t-s)A} B(u_x(s), u_x(s)) ds + \int_0^t e^{-\nu(t-s)A} \sqrt{2A} dw(s).$$

Proposition 3.3 *Let $0 < \alpha < 1$. For any $x \in H^{-\alpha}$ there exists a random time τ ($\tau > 0$ \mathbb{P} -a.s.) and a unique process u_x solving equation (15) on the time interval $[0, \tau]$ with initial value x and such that*

$$u_x \in C([0, \tau]; H^{-\alpha}) \quad \mathbb{P} - a.s.$$

Proof. We use a fixed point theorem to prove that equation (15) has a local mild solution. Set

$$z_0(t) = \int_0^t e^{-\nu(t-s)A} \sqrt{2A} dw(s).$$

We know that a.a. the paths of the process z leave in $C([0, T]; \mathbb{H})$.

We proceed pathwise. We define the mapping Ψ as

$$(\Psi u)(t) = e^{-\nu t A} x - \int_0^t e^{-\nu(t-s)A} B(u(s), u(s)) ds + z_0(t).$$

We have that $\Psi : C([0, T]; H^{-\alpha}) \rightarrow C([0, T]; H^{-\alpha})$. Indeed, $e^{-\nu t A}x$ and z_0 are in $C([0, T]; H^{-\alpha})$; we only need to deal with the second term in the r.h.s.:

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \left\| \int_0^t e^{-\nu(t-s)A} B(u(s), u(s)) \, ds \right\|_{H^{-\alpha}} \\
& \leq \sup_{0 \leq t \leq T} \int_0^t \|e^{-\nu(t-s)A} B(u(s), u(s))\|_{H^{-\alpha}} \, ds \\
& = \sup_{0 \leq t \leq T} \int_0^t \|A^{\frac{1+\alpha}{2}} e^{-\nu(t-s)A} A^{-\alpha-\frac{1}{2}} B(u(s), u(s))\|_{H^0} \, ds \\
& \leq \sup_{0 \leq t \leq T} \int_0^t \frac{c \|B(u(s), u(s))\|_{H^{-2\alpha-1}}}{(t-s)^{\frac{1+\alpha}{2}}} \, ds \quad \text{by (3)} \\
& \leq c \sup_{0 \leq t \leq T} \int_0^t \frac{\|u(s)\|_{H^{-\alpha}}^2}{(t-s)^{\frac{1+\alpha}{2}}} \, ds \quad \text{by (9)} \\
& \leq c \|u\|_{C([0, T]; H^{-\alpha})}^2 \sup_{0 \leq t \leq T} \int_0^t \frac{ds}{(t-s)^{\frac{1+\alpha}{2}}} \\
& = C_0 \|u\|_{C([0, T]; H^{-\alpha})}^2 T^{\frac{1-\alpha}{2}} \quad \text{for } \alpha < 1.
\end{aligned}$$

We have denoted by C_0 the latter constant, whereas in the previous lines we used the same notation for different constants. From now on we shall label only constants appearing in important relationships.

Hence, given x and z_0 we have

$$(34) \quad \|\Psi u\|_{L^\infty(0, T; H^{-\alpha})} \leq \|x\|_{H^{-\alpha}} + \|z_0\|_{C([0, T]; H^{-\alpha})} + C_0 \|u\|_{C([0, T]; H^{-\alpha})}^2 T^{\frac{1-\alpha}{2}}.$$

The continuity in time is proved with similar estimates.

From (34) it follows that, if $\|x\|_{H^{-\alpha}} + \|z_0\|_{C([0, T]; H^{-\alpha})} < \frac{R}{2}$ and $T^{\frac{1-\alpha}{2}} < \frac{1}{2C_0 R}$, then Ψ maps the ball of radius R of $C([0, T]; H^{-\alpha})$ into itself:

$$(35) \quad \text{for } R > 2\|x\|_{H^{-\alpha}} + 2\|z_0\|_{C([0, T]; H^{-\alpha})} \text{ and}$$

$$(36) \quad \tau < (2C_0 R)^{\frac{2}{\alpha-1}},$$

$$(37) \quad \text{then } \|\Psi u\|_{C([0, \tau]; H^{-\alpha})} < R \quad \text{if } \|u\|_{C([0, \tau]; H^{-\alpha})} < R.$$

For instance, we can choose

$$R = 3(\|x\|_{H^{-\alpha}} + \|z_0\|_{C([0, T]; H^{-\alpha})}) \quad \tau = [8C_0(\|x\|_{H^{-\alpha}} + \|z_0\|_{C([0, T]; H^{-\alpha})})]^{\frac{2}{\alpha-1}}.$$

Since we proceed pathwise, R and τ are random variables, almost surely positive and finite.

In addition we have that Ψ is a contraction mapping on the ball of radius R in $C([0, \tau]; H^{-\alpha})$ for R and τ defined above, that is

$$\begin{aligned}
& \forall x \in H^{-\alpha}, z_0 \in C([0, T]; H^{-\alpha}) \quad \exists R > 0, \gamma < 1, \tau > 0 : \\
& \quad \|\Psi u^1 - \Psi u^2\|_{C([0, \tau]; H^{-\alpha})} < \gamma \|u^1 - u^2\|_{C([0, \tau]; H^{-\alpha})} \\
& \quad \forall \|u^1\|_{C([0, \tau]; H^{-\alpha})} \leq R, \|u^2\|_{C([0, \tau]; H^{-\alpha})} \leq R.
\end{aligned}$$

To prove it, we use the bilinearity of the operator B

$$B(u^1, u^1) - B(u^2, u^2) = B(u^1, u^1 - u^2) + B(u^1 - u^2, u^2).$$

As done before, we get

$$\begin{aligned} \|\Psi u^1 - \Psi u^2\|_{C([0,\tau];H^{-\alpha})} &\leq \\ \sup_{0 \leq t \leq \tau} \int_0^t \|e^{-\nu(t-s)A} [B(u^1(s), u^1(s) - u^2(s)) + B(u^1(s) - u^2(s), u^2(s))]\|_{H^{-\alpha}} ds \\ &\leq C_0 [\|u^1\|_{C([0,\tau];H^{-\alpha})} + \|u^2\|_{C([0,\tau];H^{-\alpha})}] \|u^1 - u^2\|_{C([0,\tau];H^{-\alpha})} \tau^{\frac{1-\alpha}{2}}. \end{aligned}$$

The same choice of R and τ as in (35)-(36) provides the result with $\gamma = 2C_0 R \tau^{\frac{1-\alpha}{2}}$. \square

Given x and z_0 we have obtained the solution u_x on the time interval $[0, \tau]$. We could proceed beyond time τ ; we construct the solution u_x starting at time τ from $u_x(\tau)$. Our construction shows that if $\|u(0)\|_{H^{-\alpha}} + \|z_0\| \leq \frac{R}{2}$ then $\|u(\tau)\|_{H^{-\alpha}} \leq R$; starting from time τ the amplitude of the next time interval is smaller than τ and as usual for nonlinear equations we are not granted to cover the whole time interval $[0, T]$ by a repeated procedure.

To get a global solution we need an a priori estimate. This will be the argument of the next two sections; we need to approximate the nonlinear term B and to use the invariance of the measure μ^ν for the approximate problem. Then we recover the result for equation (15).

3.2 Finite dimensional approximation of B

For any $M \in \mathbb{N}$, let Π_M be the projection operator in H^0 defined as $\Pi_M(u_1, u_2, \dots) = (u_1, u_2, \dots, u_M, 0, 0, \dots)$.

Moreover, for $M \geq 3$ let B^M be the bilinear operator defined as

$$B^M(u, v) = \Pi_M B(\Pi_M u, \Pi_M v).$$

B^M is a bounded operator from $(\mathbb{R}^2)^M \times (\mathbb{R}^2)^M$ to $(\mathbb{R}^2)^M$. In addition we have the same result of Lemma 2.1

$$(38) \quad \|B^M(u, v)\|_{H^{-\alpha_3}} \leq c \|u\|_{H^{\alpha_1}} \|v\|_{H^{\alpha_2}} \quad \text{if } \alpha_1 + \alpha_2 + \alpha_3 = 1,$$

where the constant c is independent of M .

We have the relationship corresponding to (12):

$$(39) \quad \sum_{n=1}^M B_n^M(u, v) \cdot v_n = 0$$

The approximation problem associated to (15) is

$$(40) \quad \begin{cases} du^M(t) + [\nu A u^M(t) + B^M(u^M(t), u^M(t))] dt = \Pi_M \sqrt{2A} dw(t) \\ u^M(0) = \Pi_M x \end{cases}$$

We consider any finite time interval $[0, T]$ and set $\mu^{\nu, M} = \otimes_{n=1}^M \mu_n^\nu$.

In order to study this problem, we make precise some properties of the bilinear term B^M .

Lemma 3.4

$$(41) \quad \sup_M \int_{\mathbb{H}} \|B^M(x, x)\|_{H^{-1-\beta}}^p \mu^\nu(dx) \leq \int_{\mathbb{H}} \|B(x, x)\|_{H^{-1-\beta}}^p < \infty$$

for any $\beta > 0$.

Moreover, for any $p \geq 1$

$$(42) \quad \lim_{M \rightarrow \infty} \|B^M(u, u) - B(u, u)\|_{L^p(0, T; H^{-1-2\alpha})} = 0 \quad \text{if } u \in C([0, T]; H^{-\alpha}).$$

Proof. (41) is proved as in Proposition 2.2.

We give details for (42) in the case of the SABRA model. First,

$$B_{n,1}^M(x, x) - B_{n,1}(x, x) = \begin{cases} 0 & \text{for } n \leq M-2 \\ -ak_M(x_{M,1}x_{M+1,2} - x_{M,2}x_{M+1,1}) & \text{for } n = M-1 \\ -ak_{M+1}(x_{M+1,1}x_{M+2,2} - x_{M+1,2}x_{M+2,1}) & \text{for } n = M \\ -bk_M(x_{M-1,1}x_{M+1,2} - x_{M-1,2}x_{M+1,1}) & \text{for } n = M \\ -B_{n,1}(x, x) & \text{for } n \geq M+1 \end{cases}$$

and

$$B_{n,2}^M(x, x) - B_{n,2}(x, x) = \begin{cases} 0 & \text{for } n \leq M-2 \\ -ak_M(-x_{M,1}x_{M+1,1} - x_{M,2}x_{M+1,2}) & \text{for } n = M-1 \\ -ak_{M+1}(-x_{M+1,1}x_{M+2,1} - x_{M+1,2}x_{M+2,2}) & \text{for } n = M \\ -bk_M(-x_{M-1,1}x_{M+1,1} - x_{M-1,2}x_{M+1,2}) & \text{for } n = M \\ -B_{n,2}(x, x) & \text{for } n \geq M+1 \end{cases}$$

Therefore

$$\begin{aligned} \|B^M(x, x) - B(x, x)\|_{H^{-1-2\alpha}} &\leq |B_{M-1}^M(x, x) - B_{M-1}(x, x)|^2 k_{M-1}^{2(-1-2\alpha)} \\ &\quad + |B_M^M(x, x) - B_M(x, x)|^2 k_M^{2(-1-2\alpha)} \\ &\quad + \sum_{n=M+1}^{\infty} |B_n(x, x)|^2 k_n^{2(-1-2\alpha)}. \end{aligned}$$

Following the proof of Lemma 2.1 we get that for any $x \in H^{-\alpha}$ we have

$$\|B^M(x, x) - B(x, x)\|_{H^{-1-2\alpha}} \leq c \|(\Pi_{M-2} - I)x\|_{H^{-\alpha}}^2 \rightarrow 0 \quad \text{as } M \rightarrow \infty,$$

Hence,

$$\int_0^T \|B^M(u(s), u(s)) - B(u(s), u(s))\|_{H^{-\alpha}}^p ds \leq c \int_0^T \|\Pi_{M-2}u(s) - u(s)\|_{H^{-\alpha}}^{2p} ds.$$

Moreover,

$$\lim_{N \rightarrow \infty} \|\Pi_N u(s) - u(s)\|_{H^{-\alpha}}^{2p} = 0$$

for every s if $u \in C([0, T]; H^{-\alpha})$, and

$$\int_0^T \|\Pi_N u(s) - u(s)\|_{H^{-\alpha}}^{2p} ds \leq \int_0^T \|u(s)\|_{H^{-\alpha}}^{2p} ds \quad \forall N.$$

We conclude by dominated convergence. \square

For equation (40) we have the following standard result.

Proposition 3.5 *For each $x \in (\mathbb{R}^2)^\infty$ and M , there exists a unique strong solution to equation (40), which is a continuous and Markov process. Moreover, $\mu^{\nu, M}$ is the unique invariant measure for equation (40). Finally, there exists a unique stationary solution, which is a $\mu^{\nu, M}$ -stationary process.*

Proof. The existence and uniqueness result is standard; indeed, equation (40) is an evolution equation in the state space $(\mathbb{R}^2)^M$. In the finite dimensional case, the equation with a locally Lipschitz nonlinearity has a unique local solution, defined on a random time interval $[0, \tau] \subseteq [0, T]$ (see, e.g., [27]). The global solution, that is the solution existing on the whole time interval $[0, T]$, is shown to exist thanks to an a priori estimate obtained by Itô formula for $d\|u^M(t)\|_{H^0}^2$, using property (39):

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u^M(t)\|_{H^0}^2 \leq \|\Pi_M x\|_{H^0}^2 + t C(M)$$

where $C(M)$ is a constant quantity depending on M (see, e.g., [1] for the similar case of the 2D Navier-Stokes equation).

Moreover, for the finite dimensional equation (40) existence and uniqueness of an invariant measure hold true, since the noise is non-degenerate i.e. it acts on all the components (see, e.g., [21]). Let us prove that this unique invariant measure is indeed $\mu^{\nu, M}$. Denote by $u_x^M(t)$ the unique solution of (40) started at time 0 from $x \in (\mathbb{R}^2)^M$ and evaluated at time t . This uniquely defines a Markov semigroup $\{P_t^M\}_{t \geq 0}$:

$$(P_t^M \phi)(x) = \mathbb{E}[\phi(u_x^M(t))], \quad \phi \in B_b((\mathbb{R}^2)^M).$$

Actually, the semigroup can be defined as acting in $\mathcal{L}_{\mu^{\nu, M}}^1$, as we shall see in the following lines.

We now prove that the measure $\mu^{\nu, M}$ is an invariant measure for (40) in the sense that

$$(43) \quad \int P_t^M \phi \, d\mu^{\nu, M} = \int \phi \, d\mu^{\nu, M} \quad \forall t \geq 0, \quad \phi \in \mathcal{L}_{\mu^{\nu, M}}^1.$$

The invariance (43) is equivalent to the infinitesimal invariance

$$(44) \quad \int K^M \phi \, d\mu^{\nu, M} = 0 \quad \forall \phi \in D(K^M) \subset \mathcal{L}_{\mu^{\nu, M}}^1$$

being $K^M : D(K^M) \rightarrow \mathcal{L}_{\mu^{\nu, M}}^1$ the infinitesimal generator of the semigroup P_t^M in $\mathcal{L}_{\mu^{\nu, M}}^1$.

On the set $C_b^\infty((\mathbb{R}^2)^M)$ of infinitely differentiable functions bounded with all derivatives bounded, the operator K^M has the expression

$$(45) \quad K^M \phi(x) = \sum_{n=1}^M \sum_{j=1}^2 \left[k_n^2 \frac{\partial^2 \phi}{\partial x_{n,j}^2}(x) - B_{n,j}^M(x, x) \frac{\partial \phi}{\partial x_{n,j}}(x) - \nu k_n^2 x_{n,j} \frac{\partial \phi}{\partial x_{n,j}}(x) \right].$$

First, we have that

$$(46) \quad \int K^M \phi \, d\mu^{\nu, M} = 0 \quad \forall \phi \in C_b^\infty((\mathbb{R}^2)^M).$$

Indeed, if $\phi \in C_b^\infty((\mathbb{R}^2)^M)$ then $K^M \phi$ is $\mu^{\nu, M}$ -integrable; moreover,

$$\begin{aligned} & \int K^M \phi \, d\mu^{\nu, M} \\ &= \int \sum_{n,j} \left[k_n^2 \frac{\partial^2 \phi}{\partial x_{n,j}^2}(x) - B_{n,j}^M(x, x) \frac{\partial \phi}{\partial x_{n,j}}(x) - \nu k_n^2 x_{n,j} \frac{\partial \phi}{\partial x_{n,j}}(x) \right] \mu^{\nu, M}(dx) \end{aligned}$$

and integrating by parts

$$= -\nu \int \sum_{n,j} B_{n,j}^M(x, x) x_{n,j} \phi(x) \, \mu^{\nu, M}(dx) = 0 \quad \text{by (39).}$$

Secondly,

$$K^M \phi(x) \equiv Q^M \phi(x) - \sum_{n=1}^M \sum_{j=1}^2 B_{n,j}^M(x, x) \frac{\partial \phi}{\partial x_{n,j}}(x)$$

As done in Section 2.5 for the Ornstein–Uhlenbeck operator Q , we can prove that $C_b^\infty((\mathbb{R}^2)^M)$ is a domain of essential self-adjointness for Q^M . Using Theorem 1.2 of [30] we get that $C_b^\infty((\mathbb{R}^2)^M)$ is a core for K^M , that is $C_b^\infty((\mathbb{R}^2)^M)$ is dense in $D(K^M)$ with respect to the graph norm.

From this density result we have that (46) implies (44). \square

Now, define

$$(47) \quad v_x^M = u_x^M + (I - \Pi_M)z_x.$$

This process is a solution of

$$(48) \quad \begin{cases} dv_x^M(t) + [\nu A v_x^M(t) + B^M(v_x^M(t), v_x^M(t))]dt = \sqrt{2A} \, dw(t) \\ v_x^M(0) = x \end{cases}$$

We have

Proposition 3.6 *For each $x \in H^{-\alpha}$ and M , there exists a unique strong solution v_x^M to equation (48), which is a Markov process and for any $T > 0$ and finite*

$$v_x^M \in C([0, T]; H^{-\alpha}) \quad \mathbb{P} - a.s.$$

for any finite time interval $[0, T]$. Moreover

$$(49) \quad \sup_M \int \mathbb{E} \|v_x^M\|_{C([0, T]; H^{-\alpha})}^p \mu^\nu(dx) < \infty$$

for any $p \geq 1$.

The measure μ^ν is the unique invariant measure for equation (40), that is

$$(50) \quad \int \mathbb{E} \phi(v_x^M(t)) \, \mu^\nu(dx) = \int \phi(x) \, \mu^\nu(dx)$$

for any $t \geq 0$ and $\phi \in \mathcal{L}_{\mu^\nu}^1$.

Proof. According to Proposition 3.5 and to the results of Section 2.4, we have that for any $x \in H^{-\alpha}$ there exists a unique solution of (48) with paths in $C([0, T]; H^{-\alpha})$. This is therefore given by (47). Again we prove the uniqueness of the invariant measure dealing separately with the dynamics on the first M modes and on the remaining modes.

The invariance (50) is proved as in the proof of Proposition 3.5, since the Kolmogorov operator associated to equation (48) is

$$Q - \sum_{n=1}^M \sum_{j=1}^2 B_{n,j}^M \frac{\partial}{\partial x_{n,j}}.$$

What remains to be proved is (49). Consider the mild form of the solution to equation (48)

$$v_x^M(t) = z_x(t) - \int_0^t e^{-\nu(t-s)A} B^M(v_x^M(s), v_x^M(s)) ds$$

Then

$$(51) \quad \|v_x^M\|_{C([0,T];H^{-\alpha})}^p \leq 2^{p-1} \|z_x\|_{C([0,T];H^{-\alpha})}^p + 2^{p-1} \left(\sup_{0 \leq t \leq T} \left\| \int_0^t e^{-\nu(t-s)A} B^M(v_x^M(s), v_x^M(s)) ds \right\|_{H^{-\alpha}} \right)^p.$$

We estimate the latter term;

$$\begin{aligned} & \left\| \int_0^t e^{-\nu(t-s)A} B^M(v_x^M(s), v_x^M(s)) ds \right\|_{H^{-\alpha}} \\ & \leq \sup_{0 \leq t \leq T} \int_0^t \|e^{-\nu(t-s)A} B^M(v_x^M(s), v_x^M(s))\|_{H^{-\alpha}} ds \\ & = \sup_{0 \leq t \leq T} \int_0^t \|A^{\frac{1}{2}} e^{-\nu(t-s)A} A^{-\frac{1+\alpha}{2}} B^M(v_x^M(s), v_x^M(s))\|_{H^0} ds \\ & \leq c \sup_{0 \leq t \leq T} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|B^M(v_x^M(s), v_x^M(s))\|_{H^{-1-\alpha}} ds \quad \text{by (3)} \\ & \leq c \sup_{0 \leq t \leq T} \left(\int_0^t \frac{ds}{(t-s)^{\frac{3}{4}}} \right)^{\frac{2}{3}} \left(\int_0^t \|B^M(v_x^M(s), v_x^M(s))\|_{H^{-1-\alpha}}^3 ds \right)^{\frac{1}{3}} \quad \text{by Hölder ineq.} \\ & = c(4T^{1/4})^{\frac{2}{3}} \left(\int_0^T \|B^M(v_x^M(s), v_x^M(s))\|_{H^{-1-\alpha}}^3 ds \right)^{\frac{1}{3}}. \end{aligned}$$

Now in (51) we take the integral with respect to \mathbb{P} and μ^ν . Using the invariance of the measure μ^ν , we get

$$\begin{aligned} & \int \mathbb{E} \|v_x^M\|_{C([0,T];H^{-\alpha})}^p \mu^\nu(dx) \\ & \leq c \int \mathbb{E} \|z_x\|_{C([0,T];H^{-\alpha})}^p \mu^\nu(dx) + cT^{\frac{p}{6}} \left(T \int \|B^M(x, x)\|_{H^{-1-\alpha}}^{3p} \mu^\nu(dx) \right)^{\frac{p}{3}}. \end{aligned}$$

Finally, the uniform estimate (49) follows from the uniform estimate (41). \square

Remark 3.7 From the properties of equations (40) and (17), we also have that there exists a unique stationary solution of (48), which is a μ^ν -stationary process.

3.3 Global existence

Let us come back to the local existence result. Consider the solutions v_x^M as living in $C([0, \tau]; H^{-\alpha})$. Estimates similar to those of Section 3.1 allow to prove that the whole sequence v_x^M converges (pathwise) to u_x in the $C([0, \tau]; H^{-\alpha})$ -norm.

Proposition 3.8

$$(52) \quad \lim_{M \rightarrow \infty} \|u_x - v_x^M\|_{C([0, \tau]; H^{-\alpha})} = 0 \quad \mathbb{P} - a.s.$$

Proof. We have

$$u_x(t) - v_x^M(t) = \int_0^t e^{-\nu(t-s)A} [B^M(v_x^M(s), v_x^M(s)) - B(u_x(s), u_x(s))] ds.$$

Moreover

$$B^M(v^M, v^M) - B(u, u) = B^M(v^M, v^M - u) + B^M(v^M - u, u) + B^M(u, u) - B(u, u);$$

therefore

$$\begin{aligned} & \|u_x - v_x^M\|_{C([0, \tau]; H^{-\alpha})} \\ & \leq \sup_{0 \leq t \leq \tau} \int_0^t \|e^{-\nu(t-s)A} [B^M(v_x^M(s), v_x^M(s) - u_x(s)) + B^M(v_x^M(s) - u_x(s), u_x(s))] \|_{H^{-\alpha}} ds \\ & \quad + \sup_{0 \leq t \leq \tau} \int_0^t \|e^{-\nu(t-s)A} [B^M(u_x(s), u_x(s)) - B(u_x(s), u_x(s))] \|_{H^{-\alpha}} ds. \end{aligned}$$

We estimate

$$\|B^M(v^M, v^M - u)\|_{H^{-1-2\alpha}} + \|B^M(v^M - u, u)\|_{H^{-1-2\alpha}} \leq c(\|v^M\|_{H^{-\alpha}} + \|u\|_{H^{-\alpha}}) \|v^M - u\|_{H^{-\alpha}}$$

as done in (9). Moreover

$$\begin{aligned} & \int_0^t \|e^{-\nu(t-s)A} [B^M(u_x(s), u_x(s)) - B(u_x(s), u_x(s))] \|_{H^{-\alpha}} ds \\ & \leq \int_0^t \frac{c}{(t-s)^{\frac{1+\alpha}{2}}} \|B^M(u_x(s), u_x(s)) - B(u_x(s), u_x(s))\|_{H^{-1-2\alpha}} ds \quad \text{by (3)} \\ & \leq \left(c \int_0^t \frac{ds}{(t-s)^{\frac{1+\alpha}{1+\sqrt{\alpha}}}} \right)^{\frac{1+\sqrt{\alpha}}{2}} \left(\int_0^t \|B^M(u_x(s), u_x(s)) - B(u_x(s), u_x(s))\|_{H^{-1-2\alpha}}^{\frac{2}{1-\sqrt{\alpha}}} ds \right)^{\frac{1-\sqrt{\alpha}}{2}} \end{aligned}$$

by Hölder inequality.

With usual computations we get

$$\begin{aligned} \|u_x - v_x^M\|_{C([0, \tau]; H^{-\alpha})} & \leq C_0 \tau^{\frac{1-\alpha}{2}} (\|v_x^M\|_{C([0, \tau]; H^{-\alpha})} + \|u_x\|_{C([0, \tau]; H^{-\alpha})}) \|v_x^M - u_x\|_{C([0, \tau]; H^{-\alpha})} \\ & \quad + C_1 \tau^{\frac{\sqrt{\alpha}-\alpha}{2}} \|B^M(u_x, u_x) - B(u_x, u_x)\|_{L^{\frac{2}{1-\sqrt{\alpha}}}(0, \tau; H^{-1-2\alpha})} \\ & \leq 2C_0 R \tau^{\frac{1-\alpha}{2}} \|v^M - u\|_{C([0, \tau]; H^{-\alpha})} \\ & \quad + C_1 \tau^{\frac{\sqrt{\alpha}-\alpha}{2}} \|B^M(u_x, u_x) - B(u_x, u_x)\|_{L^{\frac{2}{1-\sqrt{\alpha}}}(0, \tau; H^{-1-2\alpha})} \end{aligned}$$

choosing R and τ as in (35)-(36). Therefore

$$\underbrace{[1 - 2C_0 R \tau^{\frac{1-\alpha}{2}}]_{>0}}_{>0} \|u_x - v_x^M\|_{C([0,\tau]; H^{-\alpha})} \leq C_1 \tau^{\frac{\sqrt{\alpha}-\alpha}{2}} \underbrace{\|B^M(u_x, u_x) - B(u_x, u_x)\|_{L^{\frac{2}{1-\sqrt{\alpha}}}([0,\tau]; H^{-1-2\alpha})}}_{\rightarrow 0 \text{ by (42)}}.$$

This implies (52). \square

On the other hand, (49) says that the sequence $\{v_x^M\}_M$ is bounded in $L^p(\Omega \times H^{-\alpha}, \mathbb{P} \otimes \mu^\nu; C([0, T]; H^{-\alpha}))$. Therefore, by the Banach-Alaoglu theorem there exists a subsequence $\{v_x^{M_i}\}_i$ \star -weakly converging in $L^p(\Omega \times H^{-\alpha}, \mathbb{P} \otimes \mu^\nu; C([0, T]; H^{-\alpha}))$ to some v_x ; moreover, the limit satisfies

$$\int \mathbb{E} \|v_x\|_{C([0,T]; H^{-\alpha})}^p \mu^\nu(dx) < \infty.$$

In particular, for μ^ν -a.e. x

$$(53) \quad \|v_x\|_{C([0,T]; H^{-\alpha})}^p < \infty \quad \mathbb{P} - a.s.$$

Since we also know that the whole sequence $\{v_x^M\}_M$ converges pathwise to u_x in $C([0, \tau]; H^{-\alpha})$ we have

$$u_x|_{[0,\tau]} = v_x|_{[0,\tau]} \quad \mathbb{P} - a.s.$$

and

$$\|u_x\|_{C([0,\tau]; H^{-\alpha})} \leq \|v_x\|_{C([0,T]; H^{-\alpha})} \quad \mathbb{P} - a.s.$$

Now, by means of this bound we get that the path u_x exists on the time interval $[0, T]$. Indeed, if we choose $\tau < (2C_0 R)^{\frac{2}{\alpha-1}}$ with $R > 2\|v_x\|_{C([0,T]; H^{-\alpha})} + 2\|z_0\|_{C([0,T]; H^{-\alpha})}$, we can repeat the construction of the path u_x on the time interval $[\tau, 2\tau]$ and so on until we recover the whole time interval $[0, T]$. This gives (31).

As far as the invariance of the measure μ^ν is concerned, notice that we proved that

$$\lim_{M \rightarrow \infty} \|u_x - v_x^M\|_{C([0,T]; H^{-\alpha})} = 0 \quad \mathbb{P} - a.s.$$

Hence, given $\phi \in C(H^{-\alpha})$

$$\lim_{M \rightarrow \infty} \phi(v_x^M(t)) = \phi(u_x(t)) \quad \mathbb{P} - a.s.$$

for any t . From (50) and by the dominated convergence theorem we get that for any $\phi \in C_b(H^{-\alpha})$

$$\int \mathbb{E} \phi(u_x(t)) \mu^\nu(dx) = \int \phi(x) \mu^\nu(dx), \quad \forall t \geq 0$$

that is

$$\int P_t \phi d\mu^\nu = \int \phi d\mu^\nu, \quad \forall t \geq 0.$$

Because of (31) we can extend this property to any $\phi \in \mathcal{L}_{\mu^\nu}^q$ ($1 \leq q < \infty$) and we get the invariance (32).

Remark 3.9 We point out that we have not proved that FC_b^∞ is a core for the infinitesimal generator of the semigroup P_t in $\mathcal{L}_{\mu^\nu}^1$, whereas we proved before (for any M) that FC_b^∞ is a core for the infinitesimal generator of the semigroup P_t^M in $\mathcal{L}_{\mu^\nu}^1$. The criterium of [30] used in the previous sections (with the quadratic term B^M) does not work for the operator K (with the "full" quadratic term B). There are few methods to prove this kind of results for infinite-dimensional non-symmetric operators; also the approximative approach of Eberle (see [19] Chapter 5) is not helpful. However, thanks to the pathwise uniqueness we have proved the strong Markov uniqueness of the Kolmogorov operator (K, FC_b^∞) in $\mathcal{L}_{\mu^\nu}^1$ (with the terminology of [19]). Actually, the result of strong Markov uniqueness would be true with respect to any invariant measure for equation (15) whose support is included in $H^{-\alpha}$ for some $0 < \alpha < 1$.

We conclude the analysis of the existence and uniqueness result, by noting that when we consider as initial data a random variable independent of w and with law μ^ν , we can construct a unique strong solution u_{st} with the same technique of Theorem 3.1; indeed the initial data $u_{st}(0)$ is such that $\|u_{st}(0)\|_{\mathbb{H}}^2 < \infty$ a.s.. This solution is the limit of the μ^ν -stationary Galerkin approximations v_{st}^M

$$\begin{cases} dv_{st}^M(t) + [\nu A v_{st}^M(t) + B^M(v_{st}^M(t), v_{st}^M(t))]dt = \sqrt{2A} dw(t), & t > 0 \\ v_{st}^M(0) \text{ has law } \mu^\nu \end{cases}$$

Therefore we get that also this solution is a μ^ν -stationary process. The only difference in the proof is that (49) is replaced by

$$\sup_M \mathbb{E} \|v_{st}^M\|_{C([0,T]; H^{-\alpha})}^p < \infty$$

and (50) by

$$\mathbb{E} \phi(v_{st}^M(t)) = \int \phi(x) \mu^\nu(dx) \quad \forall t \geq 0.$$

Hence we have

Proposition 3.10 *If $u(0)$ is a random variable with law μ^ν and independent of w , there exists a unique strong solution to equation (15) with paths in $C([0, \infty); \mathbb{H})$ \mathbb{P} -a.s.. This is a μ^ν -stationary process.*

This allows to get (32) directly.

4 Inviscid shell models

Consider equation (16). It has been studied considering initial data of finite energy or even more regular (see [15, 7, 9] and the references therein). However, we are interested in solutions having μ^ν as invariant measure. This requires to deal with initial data in the space \mathbb{H} but not in the space H^0 .

Notice that equation (16) is obtained from the viscous stochastic shell model (15) by neglecting the viscous and the stochastic terms. For this reason, for any $\varepsilon > 0$ let us consider the equation

$$(54) \quad du^\varepsilon(t) + [\nu \varepsilon A u^\varepsilon(t) + B(u^\varepsilon(t), u^\varepsilon(t))]dt = \sqrt{2\varepsilon A} dw(t), \quad t > 0.$$

For $\varepsilon = 0$ this reduces to equation (16).

To analyse equation (54) we can apply the results of the previous section; they hold true for any $\varepsilon > 0$. The fact that the measure μ^ν is an invariant measure for any $\varepsilon > 0$ can be easily checked by looking at the expression of the Kolmogorov operator associated to equation (54): $K^\varepsilon = \varepsilon Q + L$. Therefore, according to Proposition 3.10 equation (54) has a unique μ^ν -stationary solution $\bar{v}^{\nu,\varepsilon}$; this process is a strong solution and has paths in $C([0, \infty); H^{-\alpha})$ ($\alpha > 0$) a.s..

We are going to prove that there exists a subsequence $\{\bar{v}^{\nu,\varepsilon_n}\}_n$ converging in a suitable sense as $\varepsilon_n \rightarrow 0$ to a process which solves (16) for $t \geq 0$. First, we have

Proposition 4.1 *For any $0 \leq \tilde{\beta} < \frac{1}{2}$, $\tilde{\alpha} > 0$ and $T > 0$, the family $\{\bar{v}^{\nu,\varepsilon}\}_{0 < \varepsilon \leq 1}$ is tight in $C^{\tilde{\beta}}([0, T]; H^{-2-\tilde{\alpha}})$.*

Proof. We write equation (54) in the integral form:

(55)

$$\bar{v}^{\nu,\varepsilon}(t) = \bar{v}^{\nu,\varepsilon}(0) - \nu\varepsilon \int_0^t A\bar{v}^{\nu,\varepsilon}(s) ds - \int_0^t B(\bar{v}^{\nu,\varepsilon}(s), \bar{v}^{\nu,\varepsilon}(s)) ds + \sqrt{2\varepsilon}Aw(t).$$

$\bar{v}^{\nu,\varepsilon}(t)$ and $\bar{v}^{\nu,\varepsilon}(0)$ are random variables with law μ^ν . We estimate the latter three terms.

First, for μ^ν -stationary processes we have

$$\begin{aligned} \mathbb{E} \left\| \int_0^\cdot A\bar{v}^{\nu,\varepsilon}(s) ds \right\|_{W^{1,p}(0,T;H^{-2-\alpha})}^p & \\ (56) \quad & \leq (1+T^p) \int_0^T \mathbb{E}[\|\bar{v}^{\nu,\varepsilon}(s)\|_{H^{-\alpha}}^p] ds \\ & = T(1+T^p) \int_{\mathbb{H}} \|x\|_{H^{-\alpha}}^p \mu^\nu(dx) =: \tilde{C}_p \end{aligned}$$

$$\begin{aligned} \mathbb{E} \left\| \int_0^\cdot B(\bar{v}^{\nu,\varepsilon}(s), \bar{v}^{\nu,\varepsilon}(s)) ds \right\|_{W^{1,p}(0,T;H^{-1-\alpha})}^p & \\ (57) \quad & \leq (1+T^p) \int_0^T \mathbb{E}[\|B(\bar{v}^{\nu,\varepsilon}(s), \bar{v}^{\nu,\varepsilon}(s))\|_{H^{-1-\alpha}}^p] ds \\ & = T(1+T^p) \int_{\mathbb{H}} \|B(x, x)\|_{H^{-1-\alpha}}^p \mu^\nu(dx) \end{aligned}$$

Moreover, for any $0 \leq \beta < \frac{1}{2}$, $\alpha > 0$

$$(58) \quad \mathbb{E}[\|\sqrt{A}w\|_{C^\beta([0,T];H^{-1-\alpha})}] \leq \overline{C}_\beta \quad \text{by (14).}$$

Since $\varepsilon \leq 1$, we get

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E} \left\| \int_0^\cdot \nu\varepsilon A\bar{v}^{\nu,\varepsilon}(s) ds \right\|_{W^{1,p}(0,T;H^{-2-\alpha})}^p \leq \nu^p \tilde{C}_p$$

and

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E}[\|\sqrt{2\varepsilon}Aw\|_{C^\beta([0,T];H^{-1-\alpha})}] \leq \sqrt{2}\overline{C}_\beta$$

Now, we use that $W^{1,p}(0, T) \subset C^\beta([0, T])$ if $1 - \frac{1}{p} > \beta$. Then, using the previous estimates in (55), given any $0 \leq \beta < \frac{1}{2}$, $p > \frac{1}{1-\beta}$ and $\alpha > 0$ we have

$$(59) \quad \sup_{0 < \varepsilon \leq 1} \mathbb{E}[\|\bar{v}^{\nu, \varepsilon}\|_{C^\beta([0, T]; H^{-2-\alpha})}^p] < \infty.$$

On the other hand, the space $C^\beta([0, T]; H^{-1-\alpha})$ is compactly embedded in $C^{\tilde{\beta}}([0, T]; H^{-1-\tilde{\alpha}})$ if $\tilde{\alpha} > \alpha, \tilde{\beta} < \beta$; this follows from the compact embedding $H^{-1-\alpha} \Subset H^{-1-\tilde{\alpha}}$ and from the Ascoli-Arzelà theorem.

The tightness follows from (59) as usual by means of Chebyshev inequality, since α and β are arbitrary values with the stated restrictions. \square

Similarly we work on the time interval $[-T, 0]$ by considering the reversed-time parabolic nonlinear equation

$$(60) \quad du^\varepsilon(t) + [-\nu \varepsilon A u^\varepsilon(t) + B(u^\varepsilon(t), u^\varepsilon(t))]dt = \sqrt{2\varepsilon A} dw(t), \quad t < 0$$

It has a unique μ^ν -stationary solution $\underline{v}^{\nu, \varepsilon}$; this process is a strong solution and has paths in $C((-\infty, 0]; H^{-\alpha})$ ($\alpha > 0$) a.s.. Moreover, the family $\{\underline{v}^{\nu, \varepsilon}\}_{0 < \varepsilon \leq 1}$ is tight in $C^{\tilde{\beta}}([-T, 0]; H^{-2-\tilde{\alpha}})$ for any $0 \leq \tilde{\beta} < \frac{1}{2}$, $\tilde{\alpha} > 0$ and $T > 0$.

Now, we get the existence result.

Theorem 4.2 *For any $\nu > 0$, there exists a μ^ν -stationary process, whose paths solve (a.s.) equation (16) on the time interval $(-\infty, \infty)$ and are in $C^\gamma(\mathbb{R}; H^{-1-\alpha})$ for any $0 \leq \gamma < 1, \alpha > 0$.*

Proof. Let us fix $\nu > 0$. We first construct the solution for $t \geq 0$; then we can get the result for $t < 0$ with the same procedure.

By the tightness result and Prohorov theorem, the sequence of the laws of $\bar{v}^{\nu, \varepsilon}$ has a subsequence $\{\bar{v}^{\nu, \varepsilon_n}\}_{n=1}^\infty$ weakly convergent as $n \rightarrow \infty$ (with $\varepsilon_n \rightarrow 0$) in $C^\beta([0, T]; H^{-2-\alpha})$ to some limit measure. By a diagonal argument, this holds for any T and therefore the limit measure m^ν leaves in $C^\beta([0, \infty); H^{-2-\alpha})$. By Skorohod theorem, there exist a probability space $(\tilde{\Omega}^\nu, \tilde{\mathbb{F}}^\nu, \tilde{\mathbb{P}}^\nu)$, a random variable \tilde{v}^ν and a sequence $\{\tilde{v}^{\nu, \varepsilon}\}$ such that $\text{law}(\tilde{v}^{\nu, \varepsilon}) = \text{law}(\bar{v}^{\nu, \varepsilon})$, $\text{law}(\tilde{v}^\nu) = m^\nu$ and $\tilde{v}^{\nu, \varepsilon}$ converges to \tilde{v}^ν a.s. in $C^\beta([0, \infty); H^{-2-\alpha})$.

We now identify the equation satisfied by \tilde{v}^ν . We are going to prove that $\tilde{\mathbb{P}}^\nu$ -almost each path solves (16). The linear term and the stochastic term, in which appear ε and $\sqrt{\varepsilon}$ respectively, go to zero. The convergence of the nonlinear term towards $B(\tilde{v}^\nu, \tilde{v}^\nu)$ is proved by means of the bilinearity of B and by (11). We have

$$\begin{aligned} & \int_0^t \|B(\tilde{v}^{\nu, \varepsilon}(s), \tilde{v}^{\nu, \varepsilon}(s)) - B(\tilde{v}^\nu(s), \tilde{v}^\nu(s))\|_{H^{-5-2\alpha}} ds \\ & \leq \int_0^t \|B(\tilde{v}^{\nu, \varepsilon}(s), \tilde{v}^{\nu, \varepsilon}(s) - \tilde{v}^\nu(s))\|_{H^{-5-2\alpha}} ds \\ & \quad + \int_0^t \|B(\tilde{v}^{\nu, \varepsilon}(s) - \tilde{v}^\nu(s), \tilde{v}^\nu(s))\|_{H^{-5-2\alpha}} ds \\ & \leq ct \|\tilde{v}^{\nu, \varepsilon} - \tilde{v}^\nu\|_{C([0, t]; H^{-2-\alpha})} \|\tilde{v}^\nu\|_{C([0, t]; H^{-2-\alpha})}. \end{aligned}$$

The stationarity is inherited from the approximating sequence:

$$\tilde{\mathbb{E}}^\nu \phi(\tilde{v}^{\nu, \varepsilon}(t)) = \int \phi(x) \mu^\nu(dx)$$

implies

$$(61) \quad \tilde{\mathbb{E}}^\nu \phi(\tilde{v}^\nu(t)) = \int \phi(x) \mu^\nu(dx)$$

for any $t \in \mathbb{R}$ and $\phi \in C_b(H^{-2-\alpha})$.

Finally, from Proposition 2.4 the right hand side of

$$\frac{d\tilde{v}^\nu}{dt}(t) = -B(\tilde{v}^\nu(t), \tilde{v}^\nu(t))$$

belongs ($\tilde{\mathbb{P}}^\nu$ -a.s.) to $L^p_{loc}(\mathbb{R}; H^{-1-\alpha})$ for any $p \in [1, \infty)$ and $\alpha > 0$. Hence $\tilde{v}^\nu \in W^{1,p}_{loc}(\mathbb{R}; H^{-1-\alpha})$; since $W^{1,p}_{loc}([-T, T]) \subset C^\gamma([-T, T])$ for $\gamma < 1 - \frac{1}{p}$, then $\tilde{v}^\nu \in C^\gamma(\mathbb{R}; H^{-1-\alpha})$ for $\gamma < 1$. \square

Hence, the paths of the process \tilde{v}^ν define a dynamics for the inviscid shell model (16), having μ^ν as invariant measure.

References

- [1] Albeverio, S.; Cruzeiro, A. B.: Global flows with invariant (Gibbs) measures for Euler and Navier-Stokes two-dimensional fluids. *Comm. Math. Phys.* 129 (1990), no. 3, 431-444.
- [2] Albeverio, S.; Ferrario, B.: Uniqueness of solutions of the stochastic Navier-Stokes equation with invariant measure given by the enstrophy. *Ann. Probab.* 32 (2004), no. 2, 1632-1649.
- [3] Albeverio, S.; Høegh-Krohn, R.: Some Markov processes and Markov fields in quantum theory, group theory, hydrodynamics and C^* -algebras. *Stochastic integrals* (Proc. Sympos., Univ. Durham, Durham, 1980), Lecture Notes in Math. 851, Springer (1981), 497-540.
- [4] Albeverio, S.; Høegh-Krohn, R.: Stochastic methods in quantum field theory and hydrodynamics. *Phys. Rep.* 77 (1981), no. 3, 193-214.
- [5] Albeverio, S.; Høegh-Krohn, R.: Stochastic flows with stationary distribution for two-dimensional inviscid fluids, *Stoch. Proc. Appl.* 31 (1989), 1-31.
- [6] Albeverio, S.; Ribeiro de Faria, M.; Høegh-Krohn, R.: Stationary measures for the periodic Euler flow in two dimensions. *J. Statist. Phys.* 20 (1979), no. 6, 585-595.
- [7] Barbato, D.; Barsanti, M.; Bessaih, H.; Flandoli, F.: Some rigorous results on a stochastic GOY model. *J. Stat. Phys.* 125 (2006), no. 3, 677-716.
- [8] Benfatto, G.; Picco, P.; Pulvirenti, M.: On the invariant measures for the two-dimensional Euler flow. *J. Statist. Phys.* 46 (1987), no. 3-4, 729-742.
- [9] Bessaih, H.; Millet, A.: Large deviation principle and inviscid shell models. *Electron. J. Probab.* 14 (2009), no. 89, 2551-2579.

- [10] Boldrighini, C.; Frigio, S.: Equilibrium states for a plane incompressible perfect fluid. *Comm. Math. Phys.* 72 (1980), no. 1, 55-76. Erratum, *ibid.* 78 (1980/81), no. 2, 303.
- [11] Caprino, S.; De Gregorio, S.: On the statistical solutions of the two-dimensional periodic Euler equations. *Math. Methods Appl. Sci.* 7 (1985), 55-73.
- [12] Chueshov, I.; Millet, A.: Stochastic 2D hydrodynamical type systems: well posedness and large deviations. *Appl. Math. Optim.* 61 (2010), no. 3, 379-420.
- [13] Cipriano, F.: The two-dimensional Euler equation: a statistical study. *Comm. Math. Phys.* 201 (1999), no. 1, 139-154.
- [14] Constantin, P.; Levant, B.; Titi, E. S.: Analytic study of shell models of turbulence. *Phys. D* 219 (2006), no. 2, 120-141.
- [15] Constantin, P.; Levant, B.; Titi, E. S.: Regularity of inviscid shell models of turbulence. *Phys. Rev. E* (3) 75 (2007), no. 1, 016304, 10 pp.
- [16] Da Prato, G.; Debussche, A.: Two-dimensional Navier-Stokes equations driven by a space-time white noise. *J. Funct. Anal.* 196 (2002), no. 1, 180-210.
- [17] Da Prato, G.; Zabczyk, J.: *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, 1992.
- [18] Debussche, A.: The 2D-Navier-Stokes equations perturbed by a delta correlated noise. Probabilistic methods in fluids, 115-129, World Sci. Publ., River Edge, NJ, 2003.
- [19] Eberle, A.: *Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators*. Lecture Notes in Mathematics 1718; Springer-Verlag, Berlin, 1999.
- [20] Gledzer, E. B.: System of hydrodynamic type admitting two quadratic integrals of motion. *Dokl. Akad. Nauk SSSR* 209 (1973), 1046-1048 (Engl. Transl.: *Sov. Phys. Dokl.* 18 (1973), 216-217).
- [21] Khas'minskii, R. Z.: *Stochastic Stability of Differential Equations*. Nauka, Moscow, 1969 (in Russian, English translation: Sijthoff and Noordhoff, Alphen aan den Rijn, 1980).
- [22] Kuo, H. H.: *Gaussian measures in Banach spaces*. Lecture Notes in Mathematics, Vol. 463. Springer-Verlag, Berlin-New York, 1975.
- [23] Lions, P.-L.: *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models*. Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, New York, 1996.
- [24] L'vov, V. S.; Podivilov, E.; Pomyalov, A.; Procaccia, I.; Vandembroucq, D.: Improved shell model of turbulence, *Physical Review E* 58 (1998), 1811-1822.

- [25] Marchioro, C.; Pulvirenti, M.: *Mathematical Theory of Incompressible Non-viscous Fluids*. Applied Mathematical Sciences 96, Springer-Verlag, New York, 1994.
- [26] Ohkitani, K. ; Yamada, M.: Lyapunov Spectrum of a Chaotic Model of Three-Dimensional Turbulence *J. Phys. Soc. Jpn.* 56 (1987), 4210-4213.
- [27] Protter, P.: *Stochastic Integration and Differential Equations*, 2nd ed., Springer, 2004.
- [28] Reed, M.; Simon, B.: *Methods of Modern Mathematical Physics. I: Functional analysis*, Academic Press, 1972.
- [29] Reed, M.; Simon, B.: *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness*, Academic Press, 1972.
- [30] Stannat, W.: First order perturbations of Dirichlet operators: existence and uniqueness. *J. Funct. Anal.* 141 (1996), no. 1, 216-248.